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Block-finite atomic orthomodular lattices*

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Abstract

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It is shown that atomic block-finite orthomodular lattices (OMLs) belong to the class of OMLs the MacNeille completion of which is an OML. Further, it is shown that a complete block-finite OML is atomic iff the interval topology on it is Hausdorff, and that a complete (o)-continuous commutator-finite and irreducible OML is atomic. Finally, compact topological OMLs are studied and some equivalent conditions under which they are profinite (i.e., isomorphic with a direct product of finite OMLs) are found.

Introduction

Block-finite orthomodular lattices (OMLs) and commutator-finite OMLs are studied by several authors (see e.g. [2, 3, 4, 8]). In Section 1 of the present paper we show that atomic block-finite OMLs belong to the class of OMLs, the MacNeille completion of which is orthomodular. This class has not yet been completely characterized. Moreover, we show that a complete block-finite OML L is atomic iff the interval topology on L is Hausdorff. In Section 2 we prove that a complete (o)-continuous commutator-finite and irreducible OML is atomic. An example of such an OML is the OML $L(H)$ of all closed subspaces of a

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two-dimensional Hilbert space H; but the interval topology on $L(H)$ is not Hausdorff. In Section 3 we prove for block-finite OMLs the equivalence of some conditions, not equivalent in general. The main result of Section 3 is that in the class of block-finite OMLs the subclass of compact topological OMLs coincides with the subclass of profinite OMLs and those are precisely all complete (o)-continuous atomic block-finite OMLs, resp. all complete atomic OMLs only finitely many atoms of which are not in the center. Finally we show that a profinite OML is block-finite iff it is a commutator-finite.

1. Block-finite atomic OMLs

An orthomodular lattice is a lattice with 0 and 1 and with orthocomplementation which satisfies the orthomodular law ($x \leq y \Rightarrow y = x \vee (x^{\perp} \wedge y)$). For more details see [10].

It is well known that any partially ordered set P can be embedded into its *MacNeille completion* $\bar{P}$ (or completion by cuts). It has been shown (see [17] that *any complete lattice* $\bar{P}$ *into which* P *can be supremum-densely and infimum-densely embedded* (i.e., every element of $\bar{P}$ is the supremum of elements of the image of P and the infimum of elements of the image of P) *is isomorphic to the MacNeille completion of* P. For an orthoposet P the MacNeille completion is always a complete ortholattice (see [10, pp. 255–256]) in which orthocomplementation extends that of P. If L is an atomic OML then the previous observations imply that its MacNeille completion $\tilde{L}$ is an atomic and atomistic ortholattice with the same set of all atoms as $\varphi(L)$ ($\varphi : L \to \tilde{L}$ is an embedding). It is known that the MacNeille completion of an OML is not necessarily orthomodular, even if L is a modular ortholattice (see [10, p. 259]). Positive results are given by Janowitz [9] for indexed OMLs, Bruns, Greechie, Harding and Roddy showed that a variety generated by a single finite OML is closed under MacNeille completion [5]. In [14] and [12] characterizations were found of OMLs, the MacNeille completions of which are compact topological OMLs, or profinite OMLs (see also [18] and [13]). For atomic OMLs some positive results are given in [15].

In this section we show that the MacNeille completion of every block-finite atomic OML is an OML.

Recall that a nonzero element a of an ortholattice L is an *atom* if $b \leq a \Rightarrow b = 0$ or $b = a$. An ortholattice L is *atomic* if every nonzero element of L contains an atom, and L is *atomistic* if every $x \in L$ is the supremum of all atoms lying under it. Note, that an atomic ortholattice need not be atomistic, while every nonzero element of an atomic OML is the supremum of an orthogonal set of atoms [10, p. 140]. A *block* in an OML L is a maximal Boolean subalgebra of L. An OML L is called *block-finite* if there are only finitely many blocks in L. Elements $x, y \in L$ are called *compatible* if $y = (x \wedge y) \vee (x^{\perp} \wedge y)$, (written $x \leftrightarrow y$). The set $M \subset L$ is called *compatible* if $x \leftrightarrow y$ for every $x, y \in M$. Every compatible set of elements of

L is included in a block. An atom of a block of an OML L is also an atom of L. On the other hand, if L is an atomic OML then, in general, every block in L need not be atomic. For example on the atomic OML $L(H)$ of all closed linear subspaces of a complex separable infinite-dimensional Hilbert space H the range of the spectral measure corresponding to a self-adjoint operator with a simple-continuous spectrum (e.g. the 'position' or 'momentum' operator) is an atomless block of $L(H)$ [1, pp. 21, 38].

For any subset K of an OML L put $C(K) = \{y \in L \mid y \leftrightarrow x \text{ for every } x \in K\}$. Then $C(L)$ is the *center* of L.

The *interval topology* τ_i on a lattice L is the topology with the base formed by sets $L \setminus \bigcup_{i=1}^{n} [a_i, b_i]$ where $n \in \mathbb{N}$, $a_i, b_i \in L$, $a_i \leq b_i$, $i = 1, 2, \ldots, n$. It is well known that the interval topology on a Boolean algebra B is Hausdorff iff B is atomic. This assertion does not hold for OMLs. (For example, $L(H)$, where H is a finite-dimensional Hilbert space with $\dim H \geq 2$, is atomic but τ_i is not Hausdorff.)

Theorem 1.1. *Let L be an atomic and block-finite OML. Then the interval topology τ_i on L is Hausdorff.*

Proof. It suffices to show that for every $x, y \in L$, $x \neq y$ there are finitely many intervals, none of which contains both x and y and the union of which covers L. Let $A = \{a \in L \mid a \text{ is an atom of } L\}$ and $\{A_i \mid i = 1, 2, \ldots, n\}$ be the family of all maximal orthogonal sets of atoms of L. Let B_i be a block of L such that $A_i \subset B_i$, $i = 1, 2, \ldots, n$. If $y \in B_i$ then $a \leq y$ or $y \leq a^{\perp}$ for every $a \in A_i$. It follows from the fact that L is atomic and A_i is a maximal orthogonal set of atoms that B_i is atomic, $i = 1, 2, \ldots, n$ and $L \subset \bigcup_{i=1}^{n} B_i$.

Suppose that $x, y \in L$, $x \neq y$. Choose $i \in \{1, 2, \ldots, n\}$. If $x, y \in B_i$ then there exists $a_i \in B_i \cap A$ such that $a_i \leq x$ and $y \leq a_i^{\perp}$ or $a_i \leq y$ and $x \leq a_i^{\perp}$. Moreover, $B_i \subset C(\{a_i\}) = [a_i, 1] \cup [0, a_i^{\perp}]$ and none of the intervals $[a_i, 1]$ and $[0, a_i^{\perp}]$ contains both x and y. If $x \not\in B_i (y \not\in B_i)$ then there exists $a_i \in B_i \cap A$ such that $x \not\in C(\{a_i\}) (y \not\in C(\{a_i\}))$ and again $B_i \subset C(\{a_i\})$. We obtain $L = \bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} ([a_i, 1] \cup [0, a_i^{\perp}])$ and none of the intervals $[a_i, 1]$, $[0, a_i^{\perp}]$, $i = 1, 2, \ldots, n$ contains both x and y. $\square$

Theorem 1.2. *Let L be a block-finite atomic OML. Then:*

(i) *The MacNeille completion $\tilde{L}$ of L is an OML.*

(ii) *Every block in L is atomic.*

(iii) *Every block in the MacNeille completion $\tilde{L}$ of L is isomorphic to the power set lattice of a maximal orthogonal set of atoms of L.*

Proof. Suppose that L is a block-finite atomic OML and let $A = \{a \in L \mid a \text{ is an atom of } L\}$. Hence the family $\{A_1, A_2, \ldots, A_n\}$ of all maximal orthogonal sets of atoms of L is finite. Let us denote by $\tilde{L}$ the MacNeille completion of L. Then $\tilde{L}$

is an atomic and atomistic ortholattice with the same set of all atoms (we identify L with its embedding into $\tilde{L}$).

(i) For every maximal orthogonal set A_k ($k = 1, 2, \dots, n$) of atoms of L there is a block B_k of L such that $A_k \subset B_k$. Moreover, as we have shown in the proof of the Theorem 1.1, $L = \bigcup_{k=1}^{n} B_k$, and B_k is atomic. Let $x, y \in \tilde{L}$ be such that $0 \neq x \leq y$. Let $A_x = \{a \in A \mid a \leq x\}$, $A_y = \{a \in A \mid a \leq y\}$. Evidently $A_x \subset A_y$. Let the set $\mathscr{E} = \{F \subset A_y \mid F \cap A_x \neq \emptyset, F \text{ is finite}\}$ be directed by the set-inclusion. For every $F \in \mathscr{E}$ we set $x_F = \bigvee A_x \cap F$, $y_F = \bigvee F$. Since $x_F \leq y_F$ and $x_F, y_F \in L$, there are orthogonal sets F_x, F_y of atoms of L and $k \in \{1, 2, \dots, n\}$ such that $F_x \subset F_y \subset A_k \subset B_k$ and $x_F = \bigvee F_x$, $y_F = \bigvee F_y$. Thus $x_F, y_F \in B_k$. The fact that $L = \bigcup_{k=1}^{n} B_k$ implies that there exists $B_{k_0} \in \{B_1, B_2, \dots, B_n\}$ and a cofinal subset $\mathscr{E}_1 \subset \mathscr{E}$ such that $x_F, y_F \in B_{k_0}$ for every $F \in \mathscr{E}_1$. Moreover,

$$x = \bigvee_{F \in \mathscr{E}} x_F = \bigvee_{F \in \mathscr{E}_1} x_F , \qquad y = \bigvee_{F \in \mathscr{E}} y_F = \bigvee_{F \in \mathscr{E}_1} y_F .$$

Thus there exist sets $A_x^*, A_y^* \subset A_{k_0}$ such that $A_x^* \subset A_y^*$ and $x = \bigvee A_x^*$, $y = \bigvee A_y^*$. Since A_{k_0} is an orthogonal set of atoms, we have that $y = \bigvee A_x^* \vee \bigvee(A_y^* \setminus A_x^*) \leq x \vee (x^{\perp} \wedge y) \leq y$. Thus $y = x \vee (x^{\perp} \wedge y)$.

(ii) Since every $x \in \tilde{L}$ is the supremum of an orthogonal set of atoms, we have that $\tilde{L} = \bigcup_{k=1}^{n} \tilde{B}_k$, where $A_k \subset B_k \subset \tilde{B}_k$, $\tilde{B}_k$ is a block of $\tilde{L}$ and B_k is a block of L. This implies that $\tilde{L}$ is block finite (see [3, 4]). By Theorem 1.1 the interval topology τ_i on $\tilde{L}$ is Hausdorff. In a complete OML L every block B is a subcomplete sublattice and hence its intersection with an interval in L is again an interval of B and conversely. Thus for every block $B \subset \tilde{L}$ the restriction $\tau_i \cap B$ is the interval topology on B (see [16, p. 72]) and hence B is atomic and $B \cap A$ is the set of all atoms of B. We conclude that the only blocks of L (of $\tilde{L}$) are those B_k (resp. $\tilde{B}_k$) for which there exists $A_k \in \{A_1, \dots, A_n\}$ such that $A_k \subset B_k \subset \tilde{B}_k$. This proves that every block in L is atomic.

(iii) This is obvious from (ii). □

Corollary 1.3. *A complete block-finite OML L is atomic iff the interval topology τ_i on L is Hausdorff.*

Proof. If τ_i on L is Hausdorff then L is atomic (see [16, p. 75]). The converse follows from Theorem 1.1. □

2. Commutator-finite (o)-continuous OMLs

A net $(x_\alpha)_{\alpha \in \mathscr{E}}$ of elements of an OML L *(o)-converges* to an element $x \in L$ if there are nets $(u_\alpha)_{\alpha \in \mathscr{E}}, (v_\alpha)_{\alpha \in \mathscr{E}} \subset L$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for every $\alpha \in \mathscr{E}$ and $u_\alpha \uparrow x$, $v_\alpha \downarrow x$ (where $u_\alpha \uparrow x$ means that the net $(u_\alpha)_{\alpha \in \mathscr{E}}$ is nondecreasing and

$\bigvee u_\alpha = x$, the meaning of $v_\alpha \downarrow x$ is dual). The *order topology* τ_o is the strongest (finest) topology on L such that the (o)-convergence of nets implies their topological convergence. An OML L is called *(o)-continuous* if for any net $(x_\alpha)_{\alpha \in \mathscr{E}} \subset L$ and any $x, y \in L$, $x_\alpha \uparrow x$ implies that $y \wedge x_\alpha \uparrow y \wedge x$. If L is an (o)-continuous OML then for any $x_\alpha, y_\alpha, x, y \in L$ ($\alpha \in \mathscr{E}$, $\mathscr{E}$ is directed set) we have that

$$x_\alpha \xrightarrow{(o)} x \quad \text{and} \quad y_\alpha \xrightarrow{(o)} y$$

implies

$$x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y\,, \quad x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y \quad \text{and} \quad x_a^\perp \xrightarrow{(o)} x^\perp\,.$$

For any $x, y \in L$ the element

$$\mathrm{com}(x, y) = (x \vee y) \wedge (x \vee y^\perp) \wedge (x^\perp \vee y) \wedge (x^\perp \vee y^\perp)$$

is called the *commutator* of x and y. It is well known that $\mathrm{com}(x, y) = 0$ iff $x \leftrightarrow y$. An OML L is called *commutator-finite* if the set $\mathrm{Com}\, L = \{\mathrm{com}(x, y) \mid x, y \in L\}$ is finite.

We recall that for any $x \in L$ with $x \neq 0$, the *interval* $[0, x]$ *is an OML* with the lattice operations inherited from L and with the relative orthocomplementation $y \in [0, x] \to y^\perp \wedge x$. Moreover, the blocks of an interval $[0, x]$ are intersections of $[0, x]$ with blocks of L that contains x (see [10, p. 39]).

In what follows we will use the following statement from [16, p. 30]: Let L be a complete atomless OML. Then L is connected in its order topology. The proof of this fact is obtained in two steps. First it is proved that every maximal chain in L is closed and connected in the order topology τ_0 on L (analogously as for maximal chains in atomless Boolean algebras). Then the assumption that $L = A_1 \cup A_2$ where $A_1 \cap A_2 = \emptyset$ and A_1, A_2 are τ_0-closed sets implies that at least one of the maximal chains over $\{0, c, a, 1\}$ or $\{0, c, b, 1\}$ with $a \in A_1$, $b \in A_2$, $c = a \wedge b$ is disconnected, which is a contradiction.

Theorem 2.1. *Every noncentral element of a complete, (o)-continuous, commutator-finite OML L contains an atom.*

Proof. On the set $\{d_1, d_2, \ldots, d_n\}$ of all commutators of L we define a real function f by $f(d_k) = k$, $k = 1, 2, \ldots, n$ (we can assume that $n \geq 2$). Moreover, we define a family $\Phi = \{f_y \mid y \in L\}$ of real functions on L by $f_y(x) = f(\mathrm{com}(x, y))$, $x, y \in L$. The function family Φ induces a topology τ_Φ on L (see [6, p. 168]) such that for any net $(x_\alpha)_\alpha$ of elements of L and any $x \in L$ it holds

$$x_\alpha \xrightarrow{\tau_\Phi} x \quad \text{iff} \quad f_y(x_\alpha) \to f_y(x)\,, \text{ for every } y \in L\,.$$

Evidently the sets $f_y^{-1}(\{k\}) = \{x \in L \mid f_y(x) = k\}$ are clopen in τ_Φ for all $k \in \{1, 2, \ldots, n\}$ and $y \in L$. Since L is (o)-continuous, we have $\tau_\Phi \subset \tau_o$. Indeed, for any $z_\alpha, z, y \in L$ ($\alpha \in \mathscr{E}$, $\mathscr{E}$ is a directed set) we have

$$
\begin{aligned}
&z_\alpha \xrightarrow{(o)} z \quad \text{implies} \\
&\operatorname{com}(z_\alpha, y) \\
&\quad = (z_\alpha \vee y) \wedge (z_\alpha \vee y^\perp) \wedge (z_\alpha^\perp \vee y) \wedge (z_a^\perp \vee y^\perp) \\
&\quad \xrightarrow{(o)} (z \vee y) \wedge (z \vee y^\perp) \wedge (z^\perp \vee y) \wedge (z^\perp \vee y^\perp) \\
&\quad = \operatorname{com}(z, y) .
\end{aligned}
$$

Hence there exists α_0 such that for every $\alpha \geq \alpha_0$ it holds $\operatorname{com}(z_\alpha, y) = \operatorname{com}(z, y)$, i.e. $f_y(z_\alpha) = f_y(z)$. In view of the definition of τ_0 we obtain $\tau_\Phi \subset \tau_o$.

Assume that $x \in L \setminus C(L)$. Then there exists $y \in L$ such that $y \not\leftrightarrow x$ and hence $d_k = \operatorname{com}(y, x) \neq \operatorname{com}(y, 0) = d_l$. This entails that $0 \not\in f_y^{-1}(\{k\}) \cap [0, x]$ and hence $f_y^{-1}(\{k\}) \cap [0, x]$ is a clopen set in the restricted topology $\tau_o \cap [0, x]$ such that $f_y^{-1}(\{k\}) \cap [0, x] \neq [0, x]$. Thus the topology $\tau_o \cap [0, x]$ is not connected. Evidently, for every $z_\alpha, z \in [0, x]$ we have $z_\alpha \xrightarrow{(o)} z$ (in $[0, x]$) iff $z_\alpha \xrightarrow{(o)} x$ (in L). Hence $\tau_o \cap [0, x]$ is the order topology in $[0, x]$, and it is not connected.

Corollary 2.2. *Let L be a complete, (o)-continuous, commutator-finite OML with $C(L) = \{0, 1\}$. Then L is atomic.* □

Corollary 2.3. *Let L be a complete, (o)-continuous and atomless OML. The following conditions are equivalent:*

(i) *L is commutator-finite.*

(ii) *L is block-finite.*

(iii) *L is a Boolean algebra.*

Proof. (ii)$\Rightarrow$(i) has been proved in [8]. In view of Theorem 2.1 we have (i)$\Rightarrow$(iii). Evidently (iii)$\Rightarrow$(ii). □

3. Block-finite compact topological OMLs

A *topological OML* (TOML) is a pair (L, τ), where L is an OML and τ is a Hausdorff topology on L such that for any nets $(x_\alpha)_\alpha, (y_\alpha)_\alpha$ of elements of L and any $x, y \in L$, we have

$$x_\alpha \xrightarrow{\tau} x \quad \text{and} \quad y_\alpha \xrightarrow{\tau} y$$

implies

$$x_\alpha \vee y_\alpha \xrightarrow{\tau} x \vee y, \quad x_\alpha \wedge y_\alpha \xrightarrow{\tau} x \wedge y \quad \text{and} \quad x_\alpha^\perp \xrightarrow{\tau} x^\perp .$$

A TOML (L, τ) is called a *compact topological OML* (CTOML) if τ is compact. A CTOML which is the projective limit of finite OMLs with their discrete topologies is called a *profinite OML*. Tae Ho Choe and R. Greechie have shown in [18] that an OML L is profinite iff it is algebraically and topologically isomorphic to a product of finite OMLs with their discrete topologies. In [13] an example was given of a CTOML which is not profinite. We show that for block-finite OMLs no such example exists. Hence the necessary and sufficient condition for block-finite CTOML to be profinite from [18, Corollary 3, (iii)] characterizes all block-finite CTOMLs. Moreover, we prove for block-finite OMLs, the equivalence of some other conditions which are not equivalent in general. An example of a complete atomic and (o)-continuous OML (hence a complete atomic TOML with respect to the order topology) which is not a CTOML is the orthomodular lattice $L(H)$ of all closed subspaces of a finite-dimensional Hilbert space H, $\dim H \geq 2$. We shall show that there are no such examples in the family of block-finite OMLs.

Lemma 3.1. *Let L be a block-finite atomic OML and let $A = \{a \in L \mid a \text{ is an atom of } L\}$. If L is (o)-continuous then to every $b \in A$ the set $A_b = \{a \in A \mid a \not\leq b^\perp\}$ is finite.*

Proof. In view of Theorem 1.1 the interval topology τ_i in L is Hausdorff and hence $\tau_o = \tau_i$ (see [7, p. 809]). Since L is (o)-continuous, for every $b \in A$ the intervals $[b, 1]$ and $[0, b^\perp]$ are clopen sets in the order topology τ_o. Hence for every $b \in A$ there is a finite set of intervals in L such that $0 \in L \setminus \bigcup_{k=1}^n [u_k, v_k] \subset [0, b^\perp]$. Thus $L \subset [0, b^\perp] \cup \bigcup_{k=1}^n [u_k, v_k] \subset [0, b^\perp] \cup \bigcup_{k=1}^n [a_k, 1]$, where $a_k \in A$ are such that $a_k \leq u_k$, $k = 1, \ldots, n$. This implies that at most n atoms are nonorthogonal to b. $\square$

Theorem 3.2. *Let L be a block-finite atomic OML with $C(L) = \{0, 1\}$. If L is (o)-continuous then L is finite.*

Proof. By Theorem 1.2 every block of L is atomic and there is one-to-one correspondence between maximal orthogonal sets of atoms of L and blocks. Let $\{A_1, A_2, \ldots, A_n\}$ be the family of all maximal orthogonal sets of atoms of L, so that $A = \bigcup_{k=1}^n A_k$ is the set of all atoms of L. It suffices to prove that A is finite. We can assume that $L \neq \{0, 1\}$. Then the condition $C(L) = \{0, 1\}$ entails that $\bigcap_{i \in F} A_i = \emptyset$ for every $F \subset \{1, 2, \ldots, n\}$ such that $\bigcup_{i \in F} A_i = A$.

Suppose that $k \in I = \{1, 2, \ldots, n\}$ and set $\mathcal{F}_k = \{F \subset I \mid F \neq I,\ k \in F\}$. For every $F \in \mathcal{F}_k$ let $A_F = \bigcap_{i \in F} A_i \setminus \bigcup_{j \in I \setminus F} A_j$. Then $A_k = \bigcup \{A_F \mid F \in \mathcal{F}_k\}$. If for $F \in \mathcal{F}_k$ it holds $\bigcup_{i \in F} A_i = A$ then $A_F = \emptyset$. In the case $\bigcup_{i \in F} A_i \neq A$, there exists $b \in A$, $b \not\in \bigcup_{i \in F} A_i$. But then, for every $a \in A$, $a \leq b^\perp$, we have $a \in \bigcup_{j \in I \setminus F} A_j$ and thus $A_F \subset A_b = \{a \in A \mid a \not\leq b^\perp\}$ which is a finite set by Lemma 3.1. This implies that A_k is a finite set for every $k \in \{1, 2, \ldots, n\}$, hence A is finite. $\square$

Theorem 3.3. *Let L be a block-finite OML. Then the following are equivalent*:

(i) *L is a profinite OML.*

(ii) *L is a CTOML.*

(iii) *(L, τ_o) is a complete atomic TOML.*

(iv) *L is a complete atomic and (o)-continuous OML.*

(v) *L is isomorphic to a product $B \times L_0$, where B is a compact Boolean algebra and L_0 is a finite OML with discrete topology.*

Proof. By [18] every CTOML (L, τ) is complete and atomic and by [14, Theorem 2.3], $\tau = \tau_o$. Hence (i)$\Rightarrow$(ii)$\Rightarrow$(iii). Moreover, (iii)$\Rightarrow$(iv). Indeed, $x_\alpha \uparrow x$ implies $x_\alpha \wedge a \xrightarrow{\tau_o} x \wedge a$, for arbitrary elements $x_\alpha, x \in L$ and arbitrary atom a of L. Thus $x_\alpha \wedge y \uparrow x \wedge y$ for any $y \in L$.

(iv)$\Rightarrow$(v) Since L is block-finite, it follows by [2] that L is isomorphic to a direct product $B_0 \times L_1 \times \cdots \times L_n$ $(n \geq 0)$, where B_0 is a Boolean algebra and $L_1, L_2, \ldots, L_n$ are OMLs with at least two blocks each and with center $\{0, 1\}$. Evidently $B_0, L_1, \ldots, L_n$ are complete atomic and (o)-continuous. Hence B_0 is a compact (i.e. complete and atomic) Boolean algebra and L_k $(k = 1, 2, \ldots, n)$ are finite by Theorem 3.2. Hence $L_0 = L_1 \times L_2 \times \cdots \times L_n$ is a finite OML.

(v)$\Rightarrow$(i) This is obvious. $\square$

Lemma 3.4. *Let L be an atomic OML and $A = \{b \in L \mid b$ is an atom of $L\}$. If the set $A \setminus C(L)$ is finite then the MacNeille completion $\tilde{L}$ of L is a profinite and block-finite OML which is isomorphic to the product $B \times L_0$, where B is a compact Boolean algebra and L_0 is a finite OML with discrete topology.*

Proof. Let us put $B = \prod_{a \in A \cap C(L)} [0, a]$, then B is a compact Boolean algebra. Set $c = \bigvee_{a \in A \setminus C(L)} a$. If $a \in A \cap C(L)$ then $a \leq b^\perp$ for every $b \in A \setminus C(L)$ and hence $a \leq c^\perp$. This implies that $[0, c] \cap A = A \setminus C(L)$ and hence $[0, c]$ is a finite OML. Evidently $B \times [0, c]$ is a profinite and block-finite OML. We define $\varphi : L \to B \times [0, c]$ by: for every $x \in L$, $\varphi(x) = (x_1, x_2) \in B \times [0, c]$ such that $x_1 = (x \wedge a)_{a \in A \cap C(L)}$, $x_2 = x \wedge c$. Clearly φ is an embedding and $\varphi(A)$ is the set of all atoms of $B \times [0, c]$ which implies that $\varphi(L)$ is supremum-dense and meet-dense in $B \times [0, c]$. $\square$

Theorem 3.5. *Let L be a block-finite atomic OML and $A = \{a \in L \mid a$ is an atom of $L\}$. Then the following are equivalent*:

(i) *For every $b \in A$ the set $D_b = \{a \in A \mid$ there exists $\{e_1, e_2, \ldots, e_n\} \subset A$ such that $a = e_1$, $b = e_n$ and $e_i \not\leq e_{i+1}^\perp$ for $i = 1, 2, \ldots, n-1\}$ is finite (L is called* strongly almost orthogonal).

(ii) *For every $b \in A$ the set $A_b = \{a \in A \mid a \not\leq b^\perp\}$ is finite (L is called* almost orthogonal).

(iii) *For every subset $S \subset A$ and every atom $q \in \bar{S} = \{p \in A \mid$ if $a \in A$ and $a \leq b^\perp$ for every $b \in S$ then $p \leq a^\perp\}$ there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subset S$ such that $q \leq \bigvee_{k=1}^n a_k$ (L is called* strongly compactly atomistic).

(iv) *For every subset* $S \subset A$ *such that* $\vee S \in L$ *and every* $p \in A$, $p \leq \vee S$ *there exists a finite subset* $\{a_1, a_2, \ldots, a_n\} \subset S$ *such that* $p \leq \vee_{k=1}^{n} a_k$ (L *is called* compactly atomistic).

(v) $A \backslash C(L)$ *is a finite set.*

Proof. Let us denote by $\tilde{L}$ the MacNeille completion of L. In view of Theorem 1.2 we have that $\tilde{L}$ is a complete atomic OML and L and $\tilde{L}$ have the same set A of all atoms (we identify L with its embedding into $\tilde{L}$). Moreover, we have that:

(i) L is strongly almost orthogonal iff $\tilde{L}$ is a profinite OML (see [12]).

(ii) L is almost orthogonal iff $\tilde{L}$ is a CTOML (see [13]).

(iii) L is strongly compactly atomistic iff $\tilde{L}$ is an atomic order-topological OML iff $(\tilde{L}, \tau_o)$ is a complete atomic TOML (see Corollary 3.4 and Theorem 3.1 in [15]).

(iv) L is compactly atomistic iff L is atomic and (o)-continuous (see Lemma 2.2 in [15]). In view of Theorem 1.1 every block-finite atomic OML has a Hausdorff interval topology. Thus we obtain that L is compactly atomistic iff $\tilde{L}$ is a CTOML (see [12]).

(v) $A \backslash C(L)$ is a finite set iff $\tilde{L}$ is isomorphic to the product $B \times L_0$, where B is a compact Boolean algebra and L_0 is a finite OML. Indeed, since L is supremum-dense in $\tilde{L}$ we have $A \cap C(\tilde{L}) = A \cap C(L)$ and hence $A \backslash C(\tilde{L}) = A \backslash C(L)$. If $\tilde{L}$ is isomorphic to the product $B \times L_0$ then clearly $A \backslash C(\tilde{L})$ is finite. On the other hand, if $A \backslash C(L)$ is finite then $\tilde{L}$ is isomorphic to the product $B \times L_0$ by Lemma 3.4.

Now by Theorem 3.3 we obtain that all the conditions (i)–(v) are equivalent. □

Suppose that L is an atomic OML and $A = \{a \in L \mid a \text{ is an atom of } L\}$. On the set A we can define an equivalence relation $\not\gamma$ by:

> for $a, b \in A$ we have $a \not\gamma b$ if there exists a finite set $\{e_1, e_2, \ldots, e_n\} \subset A$ such that $a = e_1$, $b = e_n$ and $e_i \not\leq e_{i+1}^{\perp}$ for $i = 1, 2, \ldots, n-1$.

If all the equivalence classes are finite, i.e. if for every $b \in A$ the set $D_b = \{a \in A \mid a \not\gamma b\}$ is finite then L is evidently strongly almost orthogonal. For such OMLs the following statement holds.

Theorem 3.6. *Let* L *be a strongly almost orthogonal OML. The following conditions are equivalent*:

(*i*) L *is block finite.*

(ii) L *is commutator-finite.*

Proof. Let $\{T_i \mid i \in I\}$ be the family of all equivalence classes of the equivalence relation $\not\gamma$ defined above. It was proved in [12] that $c_i = \vee T_i$ is an atom of $C(L)$, for every $i \in I$ and $\tilde{L} = \prod_{i \in I} [0, c_i]$ is a MacNeille completion of L. If $x, y \in [0, c_i]$

then com(x, y) is the same whether computed in L or in the OML $[0, c_i]$; $i \in I$. If $[0, c_i]$ is not a Boolean algebra then there exists a nonzero commutator in $[0, c_i] \subset L \subset \tilde{L}$ (we identify L with its embedding $\varphi(L)$). Thus if, for infinitely many $i \in I$ the OML $[0, c_i]$ is not a Boolean algebra, then L and $\tilde{L}$ have infinite sets of commutators. Thus L is commutator-finite iff $\tilde{L} = \prod_{i \in I} [0, c_i] = B \times [0, c_1] \times \cdots \times [0, c_n]$. This is equivalent to the block-finite property of $\tilde{L}$ and L, since $[0, c_i]$ are finite OMLs. □

Corollary 3.7. *A profinite OML L is block-finite iff L is commutator-finite.*

Proof. This is obvious from the fact that L is profinite iff L is complete and strongly almost orthogonal (see [12]). □

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